

# Classification of electromagnetic fields in general relativity and its physical applications

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## Abstract

The simplest electromagnetic fields' (general- as well as special-relativistic) classification is formulated which is based on physically motivated ideas. According to this classification these fields can belong to three types (electric, magnetic and null), each of them being split in pure and impure subtypes. Only pure null type field propagates with the fundamental velocity  $c$ , all other fields have the propagation velocity less than that of light. The reference-frame-based methods of elimination of alternative three-fields (*e.g.*, magnetic in the electric type case) are given for pure subtypes; for pure null type the generalized Doppler effect takes place instead. All three types of impure fields are shown to be **E-B**-parallelizable. Thus such an elimination in pure non-null and parallelization in all impure cases mean transformation to the reference frame co-moving with the electromagnetic field in which the Poynting vector vanishes. The methods we propose modernizing the Rainich–Misner–Wheeler approach, also permit to construct new exact Einstein–Maxwell solutions from already known seed solutions. As examples, the Kerr–Newman and Liénard–Wiechert solutions are considered, three “new” types of rotating charged black holes (with the same Kerr–Newman geometry) are presented, and new physical effects are evaluated.

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# 1 Introduction

The main idea of this paper is to present an elementary practical classification of electromagnetic fields (see also [12]) equally applicable in both relativities and having deep physical roots. In fact, there already exist classifications of stress-energy tensors (essentially, those of Segrè and Plebański, [14, 15, 17]) and, specifically, of electromagnetic fields [18]. However, one does not encounter there a direct relation to such field properties as the Poynting vector and the velocities of propagation of concrete configurations of these fields (in particular, with respect to the reference frames co-moving with the corresponding fields when these velocities are  $< c = 1$  in the natural units used here; for a detailed discussion of the Liénard–Wiechert example see our recent article [12] and a preprint with complete simple deduction of that solution [10]). Now, our aim is also to give here natural and simple prescriptions for calculation of such physical characteristics of every concrete electromagnetic field (*cf.* [8, 16, 23]).

We consider electromagnetic fields in a vacuum (but electric currents may be present, not the media with dielectric and magnetic properties), this theory being treated here on the classical (not quantum theoretical) basis. The state of motion (in general, inhomogeneous) of the electromagnetic field, which we call its propagation, is that of a concrete (preferably, exact) solution of the system of dynamical equations (of Maxwell in special theory and Einstein–Maxwell in general theory of relativity), but not the propagation of perturbations on the background of these solutions, and not the propagation of discontinuities, which belong to other problems of the field theory not considered in this paper (their treatment is already well developed and does not need immediate revision). Any motion is, naturally, relative (that occurring with the velocity of light is also relative, at least in the sense of its direction: the light aberration effect), while motions with under-luminal velocities always permit to introduce co-moving reference frames with respect to which the fields are “at rest” everywhere in the four-dimensional region of the frame determination, which includes the requirement that redistributions of the field (“deformations”) should be “caused” by deformation, acceleration and rotation of the respective co-moving reference frame, and *vice versa*. In this connection, we remind that the physical reference frame is an idealized image of a changing with time (let us not to put this concept here more precisely) distribution and motion of observers together with their observational and measuring devices, idealized primarily in the sense to be test objects,

*i.e.* they should not practically perturb the characteristics of the objects (including fields and spacetime) in the classical (non-quantum) theory. We do not touch here upon the problem of quantum theoretical description of reference frames: this question still seems not to be adequately considered in physics, although the first step in this direction was already made by Bohr and Rosenfeld [1].

## 1.1 A preview of the paper's structure

The reader will notice that in this paper the original conclusions are enlarged with a modernized review of already known facts to make the exposition more self-sufficient. In the next section the notations and definitions used in this paper are given. In section 3 we present a condensed information on description of reference frames and its application to electromagnetic fields. The short section 4 is dedicated to the classification of these fields in terms of their two invariants,  $I_1$  and  $I_2$ . In section 5 we introduce the concept of propagation velocity of electromagnetic fields in a vacuum, and it is shown that absolute value the three-velocity of all pure null fields is equal to unity. The pure electromagnetic fields (when  $I_2 = 0$ ) are considered in section 6 yielding for the pure electric and magnetic types a simple elimination of magnetic or electric field, respectively, by the corresponding choices of reference frame (subsection 6.1), and a specific rôle of the Doppler effect (with its inevitable generalization) for the pure null type in subsection 6.2. An approach to constructing exact Einstein–Maxwell solutions in the same 4-geometry as that of any exact seed Einstein–Maxwell solution one arbitrarily would choose (with the exception of pure null fields), is developed in sections 7 and 8. Section 9 is dedicated to the treatment of impure subtype ( $I_2 \neq 0$ ) of all three types of fields leading to parallelization of electric and magnetic vectors in the adequate (canonical) reference frames. In sections 10 and 11 we consider application of the methods developed in the preceding sections to some exact solutions of the Einstein–Maxwell equations in general relativity and Maxwell's equations in special relativity: in subsection 10.1 of the well-known Kerr–Newman (KN) solution (involving, as we show, in different regions different types of electromagnetic field whose electric and magnetic vectors are always collinear and radially-directed), and in subsection 11.1, the Liénard–Wiechert solution (we show that it belongs to the pure electric type and there always exists a global co-moving with this electromagnetic field non-degenerate reference frame, so that the velocity of

propagation of this field in a vacuum is everywhere less than that of light, with the exception of the future null infinity). In subsection 10.2 “new” exact solutions with pure electric and pure magnetic fields in the standard Kerr-Newman black hole geometry are presented, together with a similar black hole with impure-null-type electromagnetic field. In subsection 11.2, it is shown that a superposition of plane harmonic electromagnetic wave and homogeneous magnetic field has strictly sub-luminal velocity of its propagation in a vacuum. In the final section 12 the obtained results are summed up and concluding remarks are given.

## 2 Mathematical preliminaries

Everything will be considered in four spacetime dimensions. We use the spacetime signature  $+, -, -, -$ , Greek indices being four-dimensional (running from 0 to 3), and Latin ones, three-dimensional, with the Einstein convention of summation over dummy indices. However, in the reference frame formalism, all indices usually are Greek, and the splitting into physical space-like and timelike objects only means that the former ones are in all free indices orthogonal to the timelike monad vector (projected onto the physical three-space of the reference frame), while the physical timelike parts represent contractions with the monad in the indices which hence become absent (a change of the root-letter notation is then advisable). The indices put into individual parentheses belong to tetrad components.

For the sake of convenience and writing and reading economy, the Cartan exterior forms formalism is frequently used. In it, the coordinated basis is the set of four covectors (1-forms)  $dx^0, \dots, dx^3$ , and the orthonormal tetrad basis similarly is  $\theta^{(0)}, \dots, \theta^{(3)}$ . Every such basis 1-form, (*e.g.*,  $dx^2$ ,  $\theta^{(3)}$ ), itself represents an individual four-dimensional covector. The exterior (wedge) product simply is a skew-symmetrized tensorial product (antisymmetrization is also denoted by Bach’s square brackets which embrace the indices, while factor  $\frac{1}{(\text{their number})!}$  is supposed to be included in this definition). It is clear that the rank of a form can be from 0 (a scalar) to 4, inclusively; all forms of higher ranks vanish identically (in  $D = 4$ ). The scalar product of four-vectors or covectors is denoted by a central dot, if these vectors are written without indices ( $A \cdot B$ ), but *with* such indices this has a wider meaning, for example,  $dx^\mu \cdot dx^\nu = g^{\mu\nu}$ : literally, this means that the scalar product of two coordinated-basis covectors equals a contravariant component of the metric

tensor with the same indices as those of these factors.

The dual conjugation in the sense of components (their indices) is denoted by an asterisk over the corresponding subindices, or under upper indices; the Hodge star stands for dual conjugation of a form written more abstractly, and is denoted by an asterisk before the form; it is convenient to have in mind that, after all, it applies to the form's basis, though this is equivalent to a similar dual conjugation of the form's components (*not both at once!*). An application of a pair of Hodge stars does not change an odd-rank form and results in the change of the sign of an even-rank form (for example, the electromagnetic field 2-form  $F$ ). By this *definition*,

$$*(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) := \frac{1}{(4-p)!} E^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_l}) \quad (2.1)$$

where

$$E_{\kappa\lambda\mu\nu} := \sqrt{-g} \epsilon_{\kappa\lambda\mu\nu}, \quad E^{\kappa\lambda\mu\nu} := -\frac{1}{\sqrt{-g}} \epsilon_{\kappa\lambda\mu\nu} \quad (2.2)$$

are covariant and contravariant components of the axial Levi-Civita tensor, and the usual Levi-Civita symbol is defined as

$$\epsilon_{\kappa\lambda\mu\nu} = \epsilon_{[\kappa\lambda\mu\nu]}, \quad \epsilon_{0123} = +1 \quad (2.3)$$

(always with the *sub*indices: this is a *symbol*, though simultaneously representing components of a contravariant axial tensor density of the weight  $-1$  and a covariant axial tensor density of the weight  $+1$ ). See some details in the beginning of the introductory chapter in [11]. Finally, coming back to a formula in the end of the second paragraph of this section, we have  $*(dx^\mu \wedge *dx^\nu) = -dx^\mu \cdot dx^\nu = -g^{\mu\nu}$ . Of course, the rôle of metric properties of spacetime is somewhat hidden in the Hodge notations, as one can see from the formulae (2.1) and (2.2).

### 3 Algebra of reference frames; applications to the electromagnetic field

The central point of our paper is the use of algebraic considerations, other applications being here only of auxiliary significance, for example, the exterior differentiation operator  $d = \theta^{(\alpha)} \nabla_{X_{(\alpha)}} \wedge \equiv dx^\alpha \nabla_{\partial_\alpha} \wedge$ .

In the physical sense, a concrete reference frame (see [11]) has only to do with a state of motion (a timelike world lines' congruence, or, equivalently, its unitary tangent vector field, the monad  $\tau$ ) of a swarm of test observers together with their test measuring devices. Moreover, one additional ingredient, the metric tensor  $g$ , is needed to construct the projector  $b := g - \tau \otimes \tau$  which at the same time serves as the (formally, four-dimensional) metric tensor on the three-dimensional local subspace orthogonal to the monad field  $\tau$ ;  $b_{\mu\nu}\tau^\nu \equiv 0$ ,  $\det b \equiv 0$ .  $b$  has the signature  $0, -, -, -$ , so that the “three-dimensional” scalar product of two vectors is

$$A \bullet B := -b_{\alpha\beta}A^\alpha B^\beta \equiv *[(\tau \wedge A) \wedge *(\tau \wedge B)] \quad (3.1)$$

where these vectors are also automatically projected onto the local subspace mentioned above. If such vectors did already belong to the subspace, they usually are boldfaced:  $\mathbf{A}^\mu = b^\mu_\nu A^\nu$ . The “three-dimensional” axial vector product of two vectors now reads

$$A \times B := *(A \wedge \tau \wedge B). \quad (3.2)$$

These algebraic operations are locally equivalent to the usual three-dimensional scalar and vector products, so we denote them by essentially the same symbols. In fact, in the complete reference frame theory we similarly use the operations of gradient, divergence and curl, but, being differential operators, they are more profoundly generalized, explicitly taking into account the characteristics of inhomogeneities of general reference frames, such as acceleration, rotation and deformation (expansion and shear) which naturally cannot be present in the algebraic treatment of geometry. The uniformity of general and old traditional notations radically simplifies the physical interpretation of general- and (in non-inertial frames) special-relativistic expressions as well as of theoretically predicted effects.

Electromagnetic fields are described with the use of the covector potential  $A = A_\alpha dx^\alpha$  and the 2-form (the field tensor)

$$F = dA = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta. \quad (3.3)$$

With respect to a given reference frame  $\tau$  (see [11]), the field tensor splits into two four-dimensional (co)vectors, electric

$$\mathbf{E}_\mu = F_{\mu\nu}\tau^\nu \iff \mathbf{E} = *(\tau \wedge *F) \quad (3.4)$$

and magnetic

$$\mathbf{B}_\mu = -F_{\mu\nu}^* \tau^\nu \iff \mathbf{B} = *(\tau \wedge F), \quad (3.5)$$

both  $\perp \tau$ , thus

$$F = \mathbf{E} \wedge \tau + *(\mathbf{B} \wedge \tau). \quad (3.6)$$

It is obvious that  $\mathbf{E}$  is a polar four-vector and  $\mathbf{B}$ , an axial four-vector, both restricted to the local physical three-subspace of the  $\tau$ -reference frame. (In Cartesian coordinates and with the corresponding inertial monad, consequently, in the Minkowskian spacetime, we have the same relations as for usual contravariant three-vectors:  $\mathbf{E}^i = F_{i0} = -F^{i0}$ ,  $\mathbf{B}^i = -\frac{1}{2}\epsilon_{ijk}F_{jk} = -\frac{1}{2}\epsilon_{ijk}F^{jk}$ .) The splitting (3.4), (3.5), (3.6) follows from the observation that the Lorentz force can be expressed as

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B})_\alpha = F_{\mu\nu} (\tau^\nu + \mathbf{v}^\nu) b_\alpha^\mu. \quad (3.7)$$

Here the three-velocity of the charged particle on which acts the Lorentz force, follows from the general definition

$$u = \overset{(\tau)}{u} (\tau + \mathbf{v}) \Rightarrow \mathbf{v} = b\left(\frac{dx}{dt}, \cdot\right) \quad (3.8)$$

where  $\overset{(\tau)}{u} = u \cdot \tau = \frac{dt}{ds} = (1 - v^2)^{-1/2}$ , while  $dt = \tau_\mu dx^\mu$  ( $\tau \cdot dx$ , that is non-total differential of the physical time along an infinitesimal displacement of the particle in spacetime), and  $u$  is its four-velocity. We have to add here important comments related to the basic concepts of both relativities, and these comments could be more transparent just with the three-velocity as an intuitively clear example. The “physical” objects (such as  $\mathbf{v}$ ,  $\mathbf{E}$ , *etc.*) belong to the section orthogonal to  $\tau$  using which these objects are introduced. Thus, already in special relativity, the velocities considered in their composition law, may exist even in three distinct sections of spacetime, while three frames are participating in the composition, and it is absolutely obvious that one cannot simply add vectors from two subspaces obtaining the third one automatically lying in the third subspace, all of them having necessary properties with respect to these respective frames. And in the composition law the three-vectors being added together, frequently are “collinear” (an absurd if they belong to non-parallel sections of spacetime). This is, of course, understandable, since Einstein himself did not realize the fact of unification of space and time into the four-dimensional manifold before the famous discovery of Minkowski in 1908 (and even during several years after this discovery).

The only two electromagnetic invariants being important in the Einstein–Maxwell theory can be easily introduced:

$$I_1 = -2 * (F \wedge *F) = F_{\mu\nu} F^{\mu\nu} = 2 (\mathbf{B}^2 - \mathbf{E}^2), \quad (3.9)$$

$$I_2 = 2 * (F \wedge F) = F_{\mu\nu}^* F^{\mu\nu} = 4 \mathbf{E} \bullet \mathbf{B}. \quad (3.10)$$

These invariants enter the following important identities:

$$F_{\mu\nu} F^{\lambda\nu} - F_{\mu\nu}^* F^{\lambda\nu} = \frac{1}{2} I_1 \delta_\mu^\lambda, \quad F_{\mu\nu}^* F^{\lambda\nu} = \frac{1}{4} I_2 \delta_\mu^\lambda. \quad (3.11)$$

In fact,  $I_2$  is an axial (pseudo-) invariant whose square behaves as a usual scalar.

The electromagnetic stress-energy tensor is [9, 11]

$$T_\mu^\nu = \frac{1}{4\pi} \left( \frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda} \delta_\mu^\nu - F_{\mu\lambda} F^{\nu\lambda} \right) = -\frac{1}{8\pi} \left( F_{\mu\lambda} F^{\nu\lambda} + F_{\mu\lambda}^* F^{\nu\lambda} \right) \quad (3.12)$$

(in Gaussian units). Its (single) contraction with arbitrary monad includes the electromagnetic energy density and Poynting vector in that frame,

$$T_\mu^\nu \tau_\nu = \frac{1}{8\pi} [(\mathbf{E}^2 + \mathbf{B}^2) \tau_\mu + 2(\mathbf{E} \times \mathbf{B})_\mu], \quad (3.13)$$

and the squared expression is (see (3.11) and *cf.* [23, 16])

$$\begin{aligned} T_\mu^\nu T_\xi^\mu \tau_\nu \tau^\xi &= \frac{1}{(8\pi)^2} [(\mathbf{E}^2 + \mathbf{B}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2] \\ &\equiv \frac{1}{(8\pi)^2} [(\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E} \bullet \mathbf{B})^2] = \frac{1}{(16\pi)^2} (I_1^2 + I_2^2). \end{aligned} \quad (3.14)$$

It is interesting that these constructions are not only scalars under transformations of coordinates, but they are also independent of the reference frame choice: the right-hand side does not involve any mention of the monad at all.

## 4 A classification of electromagnetic fields

The simple and exhaustive classification of electromagnetic fields is based on existence of only two invariants, (3.9) and (3.10), built with the field tensor  $F_{\mu\nu}$ , while all other invariants are merely algebraic functions of these two



invariants (if not vanish identically). Since  $I_2$  itself is a pseudo-invariant (axial scalar) which acquires the factor  $\text{sign}(J) := J/|J|$  under a general transformation of coordinates,  $J$  being its Jacobian, the concrete sign of  $I_2$  does not matter in our classification.

In terms of  $I_1$  the invariant classification suggests three types of fields:  $I_1 < 0$  is the electric type (the electric field dominates),  $I_1 > 0$  gives the magnetic type, and to  $I_1 = 0$ , the null type corresponds. The pseudo-invariant  $I_2$  permits to work out the classification in more detail: we get additional subtypes, impure ( $I_2 \neq 0$ ) and pure ( $I_2 = 0$ ).

Below we shall see how this classification enables us to find reference frames most adequately suitable for description of concrete electromagnetic fields and even to construct new exact solutions of Einstein–Maxwell’s equations. It also gives a natural base for straightforward physical interpretation of these fields.

## 5 Propagation of electromagnetic fields

Considering the propagation of electromagnetic field, we do not include the high-frequency limits related to field discontinuities (bicharacteristics). The Poynting vector plays an important rôle in electrodynamics having two distinct meanings: of the energy density flow and of the linear momentum density due to symmetry of the electromagnetic energy-momentum tensor (in natural units velocity is dimensionless and that of light in a vacuum is  $c = 1$ ). It is worth giving more comments on physical interpretation of the Poynting vector. It does not always describe propagation of extractable energy of the field and even a real motion (see also [19]); the exclusion is here related to the special case of static and stationary fields (whose frequency is equal to zero). Thus the Poynting vector, together with the electromagnetic energy density, determines (sometimes formally) the propagation three-velocity of electromagnetic field with respect to the reference frame in which the expression (3.13) is given. We take this velocity according to Landau and Lifshitz [7] (see the problem in p. 69) as

$$\frac{\mathbf{v}}{1 + \mathbf{v}^2} = \frac{\mathbf{E} \times \mathbf{B}}{\mathbf{E}^2 + \mathbf{B}^2} \quad (5.1)$$

(an alternative definition see in [13], p. 115, formula (312),

$$\mathbf{v} = 2 \frac{\mathbf{E} \times \mathbf{B}}{\mathbf{E}^2 + \mathbf{B}^2}, \quad (5.2)$$

but this definition is false as it can be seen from subsection 11.2 below). In the preceding pages in [7], an interesting discussion of electromagnetic invariants is worth being noted. From (3.14) and (5.1) we see that

$$0 \leq \frac{|\mathbf{v}|}{1 + \mathbf{v}^2} = \frac{1}{2} \sqrt{1 - \frac{I_1^2 + I_2^2}{4(\mathbf{E}^2 + \mathbf{B}^2)^2}} = \frac{|\mathbf{E}||\mathbf{B}|}{\mathbf{E}^2 + \mathbf{B}^2} |\sin \psi| \leq \frac{1}{2}, \quad (5.3)$$

$\psi$  being the angle between  $\mathbf{E}$  and  $\mathbf{B}$  in the strict local Euclidean sense; moreover, the function  $|\mathbf{v}|/(1 + \mathbf{v}^2)$  is everywhere monotonic. In particular, this means that the propagation of all pure null fields ( $|\mathbf{E}| = |\mathbf{B}|$ ,  $\psi = \pi/2$ ) occurs with the unit absolute value of the three-velocity, the velocity of light, and all other electromagnetic fields propagate with sub-luminal velocities which can always be made equal to zero in corresponding co-moving reference frames. This is the general-relativistic conclusion, only expressed in three-dimensional notations characteristic to the general reference frame theory.

## 6 Dealing with pure electromagnetic fields

Pure electromagnetic fields represent the simplest cases, especially in the non-null types when there always exist reference frames in which either magnetic or electric field can be easily eliminated. The pure null type requires more thorough examination involving a consideration of the Doppler effect (here, its generalized counterpart) which we have to discuss below in more detail.

### 6.1 Pure electric and magnetic type fields

Vanishing of the second invariant,  $I_2$ , means that the electromagnetic field tensor, or its dual conjugate, is a simple bivector in all reference frames (the second of two necessary and sufficient conditions is four-dimensionality of the manifold under consideration), thus

$$F = U \wedge V \text{ or } *F = P \wedge Q, \quad (6.1)$$

$U$ ,  $V$ ,  $P$ , and  $Q$  being four-(co)vectors. In the first case,

$$I_1 = 2((U \cdot U)(V \cdot V) - (U \cdot V)^2) \quad (6.2)$$

obviously is negative if one of these vectors is timelike (say,  $U$ ) and another, spacelike ( $V$ ), thus  $F$  will pertain to the pure electric type (or, similarly, for

$*F$ , to the pure magnetic type; see also an alternative case considered in subsection 11.1 when vector  $U = R$  is null). Normalizing timelike  $U$  to unity (the extra coefficient may be included in  $V$ ), we can take the normalized  $U$  as a new monad in the choice of reference frame and immediately see that in this frame the magnetic vector automatically vanishes. It remains only to show that our supposition (timelike  $U$  and spacelike  $V$ ) is sufficiently general; this can be easily proven using the substitution  $V \Rightarrow V + aU$  which does not change  $F$ . Similarly we treat the problem of eliminating the electric field in the pure magnetic case using  $*F$  in (6.1).

## 6.2 Pure null type fields and the Doppler effect

Pure null type fields have both invariants equal to zero, but the very fields remain non-trivial in any non-degenerate system of coordinates as well as in any realistic reference frame (here, in the sense of  $\mathbf{E}$  and  $\mathbf{B}$ , simultaneously), although these three-vectors do transform under changes of reference frames, and their components transform under transformations of coordinates. The monad  $\tau$  under these transformation should remain always timelike, and the Jacobian of the transformation of coordinates has to be non-zero and non- $\infty$ .

As an example we consider in this subsection a special-relativistic plane electromagnetic wave in a vacuum ( $k = \omega$ ) written in Cartesian coordinates,

$$\mathbf{E} = \{0, E \cos[\omega(x - t)], 0\}, \quad \mathbf{B} = \{0, 0, E \cos[\omega(x - t)]\}, \quad (6.3)$$

and apply to it the Lorentz transformation  $t, \mathbf{r} \Rightarrow t', \mathbf{r}'$  with the three-velocity  $\pm v$  in the positive/negative direction of the  $x$  axis using the well-known change of  $\mathbf{E}$  and  $\mathbf{B}$  under this transformation. The resulting electromagnetic field then is

$$\mathbf{E}' = \{0, E' \cos[\omega'(x' - t')], 0\}, \quad \mathbf{B}' = \{0, 0, E' \cos[\omega'(x' - t')]\} \quad (6.4)$$

where

$$E' = \sqrt{\frac{1 \mp v}{1 \pm v}} E, \quad \omega' = \sqrt{\frac{1 \mp v}{1 \pm v}} \omega. \quad (6.5)$$

It is clear that the expression of  $\omega'$  in (6.5) describes the longitudinal Doppler effect while  $E'$  gives the accompanying change of the wave intensity. Since the latter is an integral part of the longitudinal Doppler effect, we consider the complete expression (6.5) as its natural generalization; the description of transversal Doppler effect has to be generalized in the similar way.

It seems that this generalization of the Doppler effect is not encountered in physics textbooks. Nevertheless, it is generally used as an important hint in the interpretation of the well-known astrophysical phenomenon of ultrarelativistic particles' jet pairs emitted by cores of some galaxies (the jet moving away from the observer not only has lower frequency, but also correspondingly lower intensity, thus this jet sometimes escapes to be observed). There is also a static ( $\omega = 0$ ) particular case of pure null electromagnetic fields involving mutually orthogonal constant vectors  $\mathbf{E}$  and  $\mathbf{B}$  (let us call it "Cartesian case" whose cylindrically symmetric analogue is used in some experiments involving electromagnetic fields with non-zero angular momentum without a genuine rotation). Such Cartesian pure null fields manifest only intensity part of the Doppler effect since in this case  $\omega = 0 = \omega'$  in (6.5). Thus the pure null electromagnetic fields can be adjusted to any non-zero and non- $\infty$  values of their intensity and frequency (only the relation of frequency to intensity remaining constant, and if the frequency had not been equal to zero from the very beginning). A complete transformation away of initially non-trivial pure null fields is however impossible in any non-degenerate frame, representing only asymptotic and not real possibility. We considered above the case of a plane-polarized wave, but similar approach works in the circular polarization case as well.

## 7 Duality rotation and electromagnetic fields with the same spacetime geometry

Let us introduce a new electromagnetic field tensor (2-form)

$$\mathcal{F} = (k + l*)F, \quad *\mathcal{F} = (k * -l)F \quad (7.1)$$

( $*$  is the Hodge star), where  $F$  is the electromagnetic field tensor belonging to some given exact self-consistent Einstein–Maxwell solution,  $k$  and  $l$  being some scalar functions to be further determined. We now set the condition that the new field  $\mathcal{F}$  has to produce the same energy-momentum tensor which follows from the old field  $F$ . Since geometry is well determined by the energy-momentum tensor, from the Bianchi identities it then follows that the standard general relativistic Maxwell equations for both fields, old and new, will be equally satisfied if the old field has no electromagnetic sources, or the sources are localized at the singularity of the old and new fields where the standard classical theory is not applicable.

The calligraphic letters will be used for all concomitants of the new electromagnetic field. Thus

$$\mathcal{F}_{\mu\nu} = kF_{\mu\nu} + lF_{\mu\nu}^*, \quad * \mathcal{F} = kF_{\mu\nu}^* - lF_{\mu\nu}, \quad (7.2)$$

$$\mathcal{E} = k\mathbf{E} - l\mathbf{B}, \quad \mathcal{B} = k\mathbf{B} + l\mathbf{E}, \quad (7.3)$$

$$\mathcal{I}_1 = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = (k^2 - l^2)I_1 + 2klI_2, \quad \mathcal{I}_2 = \mathcal{F}_{\mu\nu}^*\mathcal{F}^{\mu\nu} = (k^2 - l^2)I_2 - 2klI_1.$$

A simple calculation yields  $\mathcal{T}_\nu^\mu = \frac{k^2+l^2}{4\pi} \left( \frac{1}{4}I_1\delta_\nu^\mu - F^{\alpha\mu}F_{\alpha\nu} \right) = (k^2 + l^2)T_\nu^\mu$  (the terms with  $I_2$  cancel automatically for arbitrary  $I_2$ ). Hence the coincidence of geometries created by the two fields is guaranteed iff

$$k = \cos \alpha, \quad l = \sin \alpha, \quad (7.4)$$

so that

$$\mathcal{I}_1 = \cos 2\alpha I_1 + \sin 2\alpha I_2, \quad \mathcal{I}_2 = \cos 2\alpha I_2 - \sin 2\alpha I_1. \quad (7.5)$$

Duality rotation, how the “transformation” (7.1) with (7.4) is now interpreted (see [8]), does not change the 4-geometry compatible with the new electromagnetic field. This geometry remains the same as that created by the old field (see also [16, 8, 23]). The “angle” (complexion)  $\alpha$  of the duality rotation is, of course, an axial scalar function of coordinates which we shall now concretely determine.

## 8 Construction of new Einstein–Maxwell solutions *via* duality rotation

First, let us see how restrictive is the duality rotation. The relations (7.5) lead to a general conclusion

$$\mathcal{I}_1^2 + \mathcal{I}_2^2 = I_1^2 + I_2^2 \quad (8.1)$$

from where it follows that the pure null property is invariant under the duality rotation and it is impossible to obtain from any other type a pure null solution using this method. Together with the considerations of subsection 6.2, this means that pure null fields sharply differ from all other electromagnetic fields.

Now, let us see if pure subtypes ( $\mathcal{I}_2 = 0 \Rightarrow \mathcal{I}_1^2 = I_1^2 + I_2^2$ ) can be obtained from the impure fields ( $I_2 \neq 0$ ). The second relation in (7.5) then yields

$$\cot 2\alpha = I_1/I_2 \quad (8.2)$$

which with the first relation gives

$$\mathcal{I}_1 = \frac{I_2}{\sin 2\alpha} = \frac{I_1}{\cos 2\alpha} = \cos 2\alpha \frac{I_1^2 + I_2^2}{I_1} = \sin 2\alpha \frac{I_1^2 + I_2^2}{I_2}. \quad (8.3)$$

(In fact, we have to perform straightforward calculations for every concrete solution  $F$  and see if  $I_2$  would be sign-definitive or not in the desired region. Though the last possibility seems to be excluded by our initial supposition, it could be, naturally, softened: the duality rotation should reduce to the identity transformation at *loci* where  $I_2$  becomes equal to zero.) From (8.2) and (8.3) we come to the following conclusions: if the new field has to be pure electric ( $\mathcal{I}_1 < 0$ ), while  $I_1 < 0$  and  $I_2 > 0$ , the “angle”  $\alpha$  has to be such that  $\sin 2\alpha < 0$  and  $\cos 2\alpha > 0$ ; for  $I_1 < 0$  and  $I_2 < 0$ ,  $\alpha$  has to give  $\sin 2\alpha > 0$  and  $\cos 2\alpha > 0$ ; for  $I_1 > 0$  and  $I_2 > 0$ , there has to be  $\sin 2\alpha < 0$  and  $\cos 2\alpha < 0$ ; for  $I_1 > 0$  and  $I_2 < 0$ ,  $\sin 2\alpha > 0$  and  $\cos 2\alpha < 0$ . Similarly, we determine the position of  $\alpha$  for the pure magnetic new field.

For the null (now impure) type of the new field ( $\mathcal{I}_1 = 0 \Rightarrow \mathcal{I}_2^2 = I_1^2 + I_2^2$ ) we have to use the first relation in (7.5) yielding

$$\tan 2\alpha = -I_1/I_2. \quad (8.4)$$

The second relation gives

$$\mathcal{I}_2 = \frac{I_2}{\cos 2\alpha} = \frac{I_1}{\sin 2\alpha}, \quad \text{etc}; \quad (8.5)$$

the procedure of determination of the position of  $\alpha$  is the same as above, here only  $\mathcal{I}_2 \neq 0$ , and both signs of  $\mathcal{I}_2$  are equally admissible. It is clear that we can perform the inverse duality rotation in all these cases (in particular, coming from the impure null to pure electric or magnetic type fields).

Thus the pure and impure electric and magnetic types form together with the impure null type a mutually “transformable” (*via* duality rotation) group of electromagnetic fields disconnected from the pure null type.

## 9 Impure electromagnetic fields: parallelizing of $\mathbf{E}$ and $\mathbf{B}$

In this section we again use the classification of electromagnetic fields in two senses: in the proper one, *i.e.* with respect to  $F$ , and, simultaneously, in the

sense of the new field  $\mathcal{F}$  introduced in (7.1), but we now look for information received from  $\mathcal{F}$  about the old field  $F$ . It is already clear that when  $F$  is impure (electric, magnetic, or null), the field  $\mathcal{F}$  can be chosen as pure (electric or magnetic in everyone of these cases). An interesting feature here is that the reference frame in which only one field (electric or magnetic in the sense of  $\mathcal{F}$ ) survives, is precisely that in which  $\mathbf{E}$  and  $\mathbf{B}$  following from  $F$  are mutually parallel, and the parallelization procedure becomes completely reduced just to determination of this canonical frame (say,  $\tau'$ ). Thus, while the relations

$$\mathcal{E} \bullet \mathcal{B} \equiv \mathcal{E}' \bullet \mathcal{B}' = 0 \quad (9.1)$$

are frame-invariant (the field  $\mathcal{F}$  is chosen as belonging to the pure subtype), the property  $\mathbf{E}' \parallel \mathbf{B}'$  is realized only in the canonical frame where either  $\mathcal{E}'$  or  $\mathcal{B}'$  vanishes.

The field  $\mathcal{F}$  may be considered as a merely auxiliary one (essentially, in special relativity where its rôle in generation of gravitational field is neglected), so that the parallelization procedure then may be managed even without the use of the strict duality rotation with the “angle”  $\alpha$  and the relation (8.1), but when we simply take, *e.g.*,

$$\mathcal{F} = (1 + k*)F. \quad (9.2)$$

The calculations following from this ansatz are simple, but somewhat cumbersome, and we omit them, especially since they will not be used in this paper.

## 10 Examples in general relativity

In this section we consider two particular electromagnetic fields self-consistently sharing one and the same four-dimensional geometry: in the first subsection, the standard Kerr–Newman (KN) rotating charged black hole, and in the subsection 10.2, its generalizations to the black holes created by specific mixtures of electric charge and magnetic monopole distributions rotating as the KN singular ring. The first example corresponds to an impure electromagnetic field, while the next three ones belong to the pure electric, pure magnetic and impure null types, thus representing new black-hole exact solutions of Einstein–Maxwell equations. The first two new solutions admit reference frames in which there is no magnetic or no electric fields in the whole spacetime.

## 10.1 The Kerr–Newman solution

The KN metric tensor is taken in the Boyer–Lindquist (BL) coordinates as

$$ds^2 = \frac{\Delta}{\rho^2} (dt - a \sin^2 \vartheta d\varphi)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\vartheta^2 - \frac{\sin^2 \vartheta}{\rho^2} [(r^2 + a^2)d\varphi - a dt]^2 \quad (10.1)$$

where  $\rho^2 = r^2 + a^2 \cos^2 \vartheta$ ,  $\Delta = r^2 - 2Mr + Q^2 + a^2$ . Thus the orthonormal 1-form basis outside the singularity  $\rho = 0$  reads

$$\left. \begin{aligned} \theta^{(0)} &= \frac{\sqrt{\Delta}}{\rho} (dt - a \sin^2 \vartheta d\varphi), & \theta^{(1)} &= \frac{\rho}{\sqrt{\Delta}} dr, \\ \theta^{(2)} &= \rho d\vartheta, & \theta^{(3)} &= \frac{\sin \vartheta}{\rho} [(r^2 + a^2)d\varphi - a dt], \end{aligned} \right\} \quad (10.2)$$

so that  $d\varphi = \frac{a}{\rho\sqrt{\Delta}}\theta^{(0)} + \frac{1}{\rho\sin\vartheta}\theta^{(3)}$ . Further, the electromagnetic 1-form four-potential (*cf.* the usual Coulomb potential) and the field tensor (2-form) are

$$\left. \begin{aligned} A &\equiv A_{(\alpha)}\theta^{(\alpha)} = \frac{Qr}{\rho\sqrt{\Delta}}\theta^{(0)} \quad \text{and} \\ F &\equiv \frac{1}{2}F_{(\alpha)(\beta)}\theta^{(\alpha)} \wedge \theta^{(\beta)} = dA \\ &= \frac{Q}{\rho^4} [(r^2 - a^2 \cos^2 \vartheta)\theta^{(0)} \wedge \theta^{(1)} - 2ar \cos \vartheta \theta^{(2)} \wedge \theta^{(3)}], \end{aligned} \right\} \quad (10.3)$$

respectively (see for some details [2, 3, 4, 5, 6, 17, 20, 22]). Since

$$4a^2 r^2 \cos^2 \vartheta + (r^2 - a^2 \cos^2 \vartheta)^2 = \rho^4 \quad (10.4)$$

(this confirms the presence in  $F$  of the factor  $r^2 - a^2 \cos^2 \vartheta$  which first could seem to be somewhat unnatural), we can now introduce an “angle”  $\beta$  as

$$\sin \beta = \frac{2ar \cos \vartheta}{\rho^2}, \quad \cos \beta = \frac{r^2 - a^2 \cos^2 \vartheta}{\rho^2}. \quad (10.5)$$

Then the electromagnetic field  $F$  reads

$$F = \frac{Q}{\rho^2} (\cos \beta + \sin \beta *) (\theta^{(0)} \wedge \theta^{(1)}), \quad (10.6)$$



obviously involving a duality rotation, and the electromagnetic invariants read

$$I_1 = -\frac{2Q^2}{\rho^4} \cos 2\beta, \quad I_2 = \frac{2Q^2}{\rho^4} \sin 2\beta, \quad (10.7)$$

so that the construction invariant under duality rotation (8.1) in the KN solution case is (*cf.* the Coulomb and Reissner–Nordström fields where, of course,  $a = 0$ )

$$I_1^2 + I_2^2 = \frac{4Q^4}{\rho^8}. \quad (10.8)$$

From (10.7) and (10.5) we immediately find that on the “plane”  $\cos \vartheta = 0$  and the “sphere”  $r = 0$  (it is well known that the “negative region of space” with  $r < 0$  makes a certain sense in this spacetime) the invariants take values

$$I_1 = -\frac{2Q^2}{r^4}, \quad I_2 = 0 \quad \text{and} \quad I_1 = -\frac{2Q^2}{a^4 \cos^4 \vartheta}, \quad I_2 = 0,$$

respectively, thus the field  $F$  belongs there to the pure electric type (in the BL frame, the magnetic field is already eliminated in this region). The intersection of these 2-surfaces is the well-known singular rotating Kerr ring where  $I_2$  vanishes when we approach to the ring from these mutually orthogonal directions. This would be in conformity with the usual interpretation of the ring as rotating with the velocity of light if that  $I_1$  also tended there to zero, though this is not quite the case. The vanishing of  $I_1$  occurs only in the limits in four directions along which  $r = \epsilon(3 \pm 2\sqrt{2})a^2 \cos^2 \vartheta$  where  $\epsilon = \pm 1$  (without admission to simultaneously take only both similar signs in the whole formula). Further, when  $r^2 = a^2 \cos^2 \vartheta$ , we have the pure magnetic type field (with already eliminated electric field in the BL frame) with  $I_1 = \frac{3Q^2}{4a^4 \cos^4 \vartheta} = \frac{3Q^2}{4r^4}$ , thus, if we come to the ring from the corresponding directions, the field will be purely magnetic. The electromagnetic field around the Kerr ring in KN solution is in fact very diverse, like a patchwork quilt.

To find how behaves the propagation velocity  $\mathbf{v}$  of this field, we have to calculate its energy density and Poynting vector, but let us begin with  $\mathbf{E}$  and  $\mathbf{B}$  in the BL frame  $\tau = \theta^{(0)}$ , see (10.2). Already before any calculations, only looking at the definitions (3.4) and (3.5), one understands that these two vectors and  $\theta^{(1)}$  are everywhere collinear, so that the Poynting vector identically vanishes, as well as  $\mathbf{v}$  does (this is natural, since both metric coefficients and the field tensor components do not depend on  $t$  and  $\varphi$ , and we already *are* in the KN-field’s co-moving frame). The pure subtype can be

realized (if we suppress duality rotations) only due to local vanishing of  $\mathbf{E}$  or  $\mathbf{B}$  in the BL frame. In this co-moving frame the electromagnetic energy density is everywhere equal to

$$T^\nu_\mu \tau^\mu \tau_\nu = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \frac{1}{16\pi} \sqrt{I_1^2 + I_2^2} = \frac{Q^2}{8\pi\rho^4}, \quad (10.9)$$

see (5.3).

We have found here that the KN solution is “anisotropic” and “inhomogeneous” in the sense of distribution of electric and magnetic types of the  $F$  field, although three-vector fields  $\mathbf{E}$  and  $\mathbf{B}$  are always collinear in the spacetime of KN black hole in the BL frame, but there are surfaces on which either magnetic or electric field vanishes. In order to better understand if this structure of electromagnetic field belonging to the KN solution is inevitable (however, see also [3]), we shall further apply to the KN solution the method developed in sections 7 and 8 (as a broadening and modernization of the Rainich–Misner–Wheeler duality rotation approach). We shall find that it is easy to radically modify the KN solution obtaining rotating black holes with electromagnetic fields of everywhere pure electric or pure magnetic, or impure null type.

## 10.2 “New” rotating black hole solutions with pure electric, pure magnetic, and impure null fields $F$ in KN geometry

Here we apply duality rotation

$$\mathcal{F} = (\cos \alpha + \sin \alpha *) F, \text{ or } * \mathcal{F} = (\cos \alpha * - \sin \alpha) F \quad (10.10)$$

[the combination of (7.1) and (7.4)] to the KN electromagnetic field (10.3). Since our aim is to construct a pure subtype field ( $\mathcal{I}_2 = 0$ ), we already have from (8.1)  $\mathcal{I}_1^2 = I_1^2 + I_2^2$  while the sign of  $\mathcal{I}_1$  is determined by the choice of position of  $\alpha$  in the angle diagram and (8.2). Taking the expression (10.6) and using the obvious algebra of duality rotation, we rewrite the expression (10.10) as

$$\mathcal{F} = \frac{Q}{\rho^2} [\cos(\alpha + \beta) + \sin(\alpha + \beta) *] (\theta^{(0)} \wedge \theta^{(1)}). \quad (10.11)$$

Now, putting  $\alpha + \beta = 0$ , we immediately come to the pure electric field  $\mathcal{F}_{\text{el}}$

$$\mathcal{F}_{\text{el}} = \frac{Q}{\rho^2} \theta^{(0)} \wedge \theta^{(1)} \quad (10.12)$$

generating the same KN geometry which everybody associates with the KN “patchwork” electromagnetic field  $F$  (10.3). The pure magnetic field

$$\mathcal{F}_{\text{mag}} = \frac{Q}{\rho^2} \theta^{(2)} \wedge \theta^{(3)} \quad (10.13)$$

is similarly obtained with the use of  $\alpha + \beta = -\pi/2$ .

Finally, we have to add the third new case of KN-like black holes, those with null type electromagnetic field when in (10.11)  $\alpha + \beta = \pi/4 + n\pi/2$  is taken (naturally, this field now belongs to the impure subtype). With  $n = 0$ , it reads

$$\mathcal{F}_{\text{null}} = \frac{Q}{\sqrt{2}\rho^2} (\theta^{(0)} \wedge \theta^{(1)} - \theta^{(2)} \wedge \theta^{(3)}) . \quad (10.14)$$

The (contracted) Bianchi identities guarantee satisfaction of Maxwell’s equations outside the ring singularity for the new field  $\mathcal{F}$  in this geometry, while the presence of magnetic monopole distribution existing here only on the singular Kerr ring should not create any problem since at the singularity the classical laws of physics obviously fail to work. One may say that the magnetic monopole distribution (as well as that of the electric charge) can, as this is shown above, exactly compensate the magnetic (electric) field created by a rotating charge (rotating monopole) distribution, but this is not precisely the case. In fact, we encounter in this situation a more complicated superposition of dynamical and kinematic effects, since the  $(\tau = \theta^{(0)})$ -reference frame is not an inertial one: it involves both acceleration and rotation (the latter is present due to  $\theta^{(0)} \wedge d\theta^{(0)} \neq 0$ , see the definitions in [11]). In the same ref. 11 (chapter 4, pp. 86, 87, 90, and 91) it is shown that in the classical Maxwell equations and in laws of motion of electric charges, both written in non-inertial reference frames, there appear kinematic terms of the monopole nature. In equations of motion they bear the name of kinematic forces (forces of inertia), thus in the field equations let us speak of kinematic sources. While dynamical force and source are originated by the same interaction term in the action integral (only the variational procedure is performed with respect to particle’s world line and to field’s potential, correspondingly), their kinematic counterparts automatically appear in the respective dynamical equations written (and experimentally investigable) in non-inertial frames. It is interesting that kinematic and dynamical counterparts of forces, as well as of sources, have rather similar structure, despite their different origin, thus making them recognizable.

## 11 Examples in special relativity

### 11.1 Liénard–Wiechert’s field: the pure electric type

The Liénard–Wiechert (LW) field is special relativistic electromagnetic field generated by an arbitrarily moving electric charge  $Q$  (we restrict our consideration to an arbitrary timelike world line of the charge). See the details of deduction of this field in [10] where we used the future light cone (the case of retarded field, precisely like in the present paper); an arbitrary mixture of retarded and advanced fields can be found in [18]. Thus the retarded point on the charge world line  $x'^\mu$  is connected with the four-dimensional point  $x^\mu$  (where the field is determined) by the null vector  $R^\mu = x^\mu - x'^\mu$  (we choose for simplicity the Cartesian coordinates in this special relativistic treatment). The four-potential then reads

$$A^\mu = \frac{Qu'^\mu}{D} \quad (11.1)$$

where  $u'^\mu = dx'^\mu/ds'$  ( $u' \cdot u' = 1$ ) is the retarded four-velocity of the charge and  $D = u' \cdot R \equiv \sqrt{-\mathbf{D}^\mu \mathbf{D}_\mu}$ , while  $\mathbf{D}^\mu = R^\nu b^\mu_\nu$ ,  $b = g - u' \otimes u'$  simultaneously being the projector on local retarded three-dimensional subspace orthogonal to  $u'$  and the spatial three-metric on this subspace (with the signature  $0, -, -, -$ ). The retarded four-acceleration of the charge is  $a'^\mu = du'^\mu/ds'$  (naturally,  $a' \cdot u' \equiv 0$ ). A simple calculation yields the LW field tensor

$$F = R \wedge V, \quad V = \frac{Q}{D^2} \left( a' + u' \frac{1 - a' \cdot R}{D} \right) \quad (11.2)$$

(the second field invariant  $I_2$  automatically vanishes). The first invariant is

$$I_1 = -\frac{2Q^2}{D^4} < 0 \quad (11.3)$$

(remarkably, its structure is exactly Coulombian). Thus the LW field pertains to the pure electric type everywhere outside the point charge’s world line.

Combining  $R$  with  $V$  in (11.2), one can change the null vector  $R$  to a timelike one,  $U$ , and thus reduce the problem to that discussed in subsection 6.1. However it is much simpler to (algebraically) regauge the vector  $V \rightarrow W = V + \frac{Q}{D^2} l R$  (the fractional coefficient is put only for convenience, and  $l$  is a scalar function still to be determined); this does neither change

the field tensor,  $F = R \wedge W$ , nor produce any  $l^2$ -term in further calculations. Applying now the 1-form definition of the magnetic vector in a  $\tau$ -frame (3.5) and taking the monad as  $\tau = NW$  where the scalar normalization factor is  $N = (W \cdot W)^{-1/2}$ , we come to  $\mathbf{B} = 0$  in this frame. The problem is thus reduced to a proper choice of  $l$  such that  $W$  will be a suitable real timelike vector. We see that

$$W = \frac{Q}{D^2} \left( a' + \frac{1 - a' \cdot R}{D} u' + lR \right). \quad (11.4)$$

Then its square takes an unexpectedly simple form

$$W \cdot W = \left( \frac{Q}{D^2} \right)^2 \left[ a' \cdot a' + \frac{(1 - a' \cdot R)^2}{D^2} + 2l \right]. \quad (11.5)$$

In fact,  $l$  still remains arbitrary (this means that there is a continuum of such different co-moving frames). Let it be

$$l = \frac{1}{2} \left[ \frac{1}{D^2} - a' \cdot a' - \frac{(1 - a' \cdot R)^2}{D^2} \right] \quad (11.6)$$

(the first term in the square brackets,  $1/D^2$ , got its denominator to fit the dimensional considerations). Finally,  $W \cdot W = \left( \frac{Q}{D^3} \right)^2 > 0$  and

$$\tau = Da' + (1 - a' \cdot R) u' + \frac{1}{2D} \left[ 1 - D^2 a' \cdot a' - (1 - a' \cdot R)^2 \right] R \quad (11.7)$$

(it is clear that  $\tau \cdot \tau = +1$ ). By its definition, the monad  $\tau$  describes the reference frame co-moving with the LW electromagnetic field: in this frame the Poynting vector of the field vanishes, the electromagnetic energy flux ceases to exist due to the absence of magnetic part  $\mathbf{B}^\tau$  of the field in this frame (applicable at any finite distance  $D$ , not asymptotically), and  $F$  can be rewritten as  $F = \frac{Q}{D^3} R \wedge \tau$ . The expression (3.4) now yields

$$\mathbf{E}^\tau = *(\tau \wedge *F) = \frac{Q}{D^3} *[\tau \wedge *(R \wedge \tau)] = \frac{Q}{D^2} \mathbf{n}^\tau \quad (11.8)$$

which is, up to an understandable reinterpretation of notations, exactly the form known as the Coulomb field vector. Here the unitary vector  $\mathbf{n}^\tau = \mathbf{D}^\tau / D$  is normal to the  $\tau$ -congruence, while  $R \cdot u' = \frac{u'}{D} = D = \frac{\tau}{D} = R \cdot \tau$  and

$\overset{\tau}{\mathbf{D}}^\mu = b_\nu^\mu R^\nu$  with  $b_\nu^\mu = \delta_\nu^\mu - \tau^\mu \tau_\nu$ , hence in the frame co-moving with the LW field (the reader may choose other co-moving reference frames taking different  $l$ s, but our choice seems to be one of the simplest ones)

$$\overset{\tau}{\mathbf{D}} = -D^2 a' - D(1 - a' \cdot R) u' + \frac{1}{2} \left[ 1 + D^2 a' \cdot a' + (1 - a' \cdot R)^2 \right] R, \quad (11.9)$$

so that this  $\overset{\tau}{\mathbf{D}} \neq \overset{u'}{\mathbf{D}}$  in fact given after the formula (11.1) as  $\mathbf{D}$  and pertaining to another frame (co-moving with the retarded charge, not with its field, see for details [10]).

The situation discovered in this subsection can be formulated in a short and exact form as existence in all spacetime outside the world line of the charge generating LW's field, of a reference frame co-moving with this field, *i.e.*, a frame in which the Poynting vector vanishes in all this region (with the exception of the future null infinity which can be described only asymptotically, using more topological<sup>1</sup> than geometrical methods), thus in this frame there is no flow of electromagnetic energy anywhere. Of course, this frame is in general a rotating one (see in [10] the expression (4.27) and appendix A), thus the three-dimensional space is non-holonom (it does not form a global — at least, finite — three-dimensional subspace of the four-dimensional world; at most, in the presence of rotation there exist only strictly local (infinitesimal) elements of such a subspace which do not merge into a finite hypersurface, like scales of a sick fish in aquarium. Note that this occurs here even in the special relativity, not only in general theory. Moreover, the presence of the frame's rotation does not permit synchronization of clocks being at rest in such a frame. (In the same spacetime there always exist also an infinite number of non-rotating frames in which you are welcomed to perform a synchronization, but in any rotating frame this very procedure is strictly forbidden. It is curious that while we live all our lives in our terrestrial rotating frame, its rotation remains sufficiently slow not to condition us to this non-holonom psychology.)

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<sup>1</sup>It was a gibe of the fate with respect to the authors who deliberately disregarded topology and nevertheless claimed that the LW field contains electromagnetic radiation, though in this field the Poynting vector can be easily transformed away by means of a proper choice of the reference frame in every finite region around the charge's world line. About the attitude of L.D. Landau and E.M. Lifshitz toward topology see [21], pp. 470-471, — and see also [7], p. 173 ff. However, Misner and Wheeler *did* take topology into account in their general-relativistic considerations [8, 23].

Since everywhere outside the LW source (the pointlike charge world line on which the field singularity occurs) there is no magnetic field in this frame, and any redistribution of electromagnetic field cannot take place there, the LW field does not propagate in this frame, it only can be compressed or rarefied remaining at rest with respect to the frame in its contraction or expansion, similar to effects known in relativistic cosmology. This could seem to be in contradiction with the traditional decomposition of LW field into the near (induction) and distant (radiation) zones. The reason for this “contradiction” should be seen in the fact that, although the very Maxwell equations are linear, the physical characteristics of electromagnetic fields such as their energy density, Poynting vector (describing, up to a constant factor, either the energy flow density or linear momentum density of the field), and stress, are quadratic (or bilinear) in the field tensor sense. Thus we have not to overlook the “interaction” terms between the induction and radiation counterparts in these characteristics. Note that the elimination of the magnetic part of LW’s field is directly related to its *pure* electric type,— consequently, to the quadratic (bilinear) invariants of this field. Therefore the “contradiction” exists only in a wrong customary application of the linearity concept to the strictly nonlinear characteristics even of the electrovacuum electromagnetic fields. In a certain sense, there should be a way to reconcile this contradiction considering the asymptotic behaviour of the field; in any case, this has to correspond to merely technical details of the problem. Similarly, in his ironic paper [19], Synge with his great wit criticized the existing style of introduction of these same characteristics in the most widely used textbooks on field theory. Though his criticism was there somewhat superficial, we find Synge’s paper quite provocative in more profound determination of the quotidian concepts in our theory using their physical sense.

## 11.2 Propagation of a plane electromagnetic wave on the background of homogeneous magnetic field in a vacuum

Finally, let us consider a simple, but not yet discussed in literature problem of electromagnetic waves’ propagation in a time-independent sourceless Maxwell field in a vacuum. For simplicity, we take the same wave as in subsection 6.2, and the additional Maxwell field is chosen merely as a magnetic one in the direction of propagation of the wave, with the constant three-

vector  $\mathbf{B}$ . Obviously, this superposition is an exact solution of Maxwell's equations, and there cannot be any real interaction between these two fields since the equations are linear. We have however already noted that the velocity of propagation of electromagnetic field is non-linear in terms of this field's tensor  $F$ , so that there should, naturally, exist an observable physical effect in the case of a superposition of such free Maxwell's fields. There is, of course, an effect which was already considered and observed in the early history of optics, that of the standing electromagnetic waves, but nobody still worried about the seemingly absurd problem formulated above.

Thus we take in Cartesian coordinates  $t, x, y, z$  the superposition of the fields (6.3) and  $\mathbf{B}_H = Hdx$ , *i.e.*

$$\mathbf{E} = E \cos[\omega(x - t)]dy, \quad \mathbf{B} = Hdx + E \cos[\omega(x - t)]dz. \quad (11.10)$$

Obviously,  $I_2 = 0$  due to the orthogonality of  $\mathbf{E}$  and  $\mathbf{B}$ , and the first invariant is  $I_1 = 2H^2 > 0$ : this is the pure magnetic type field. The result of superposition (11.10) is in fact a specific not precisely monochromatic wave whose behaviour can be best understood in the reference frame co-moving with it, and one can find such a frame using the pure-magnetic property of this wave's field, see subsection 6.1. First, we write the field  $*F$  [through (11.10) in the initial frame  $\tau_{\text{in}} = dt$ ] as a simple bivector:

$$\begin{aligned} *F &= *(\mathbf{E} \wedge dt) - \mathbf{B} \wedge dt \\ &= - (E \cos[\omega(x - t)]dx \wedge dz + Hdx \wedge dt + E \cos[\omega(x - t)]dz \wedge dt) \\ &= - (Hdx + E \cos[\omega(x - t)]dz) \wedge (dt - dx) = -P \wedge Q. \end{aligned} \quad (11.11)$$

If to  $P$  we add  $lQ$  ( $l$  being an arbitrary function) and use this sum  $P'$  instead of the former  $P$ ,  $*F$  does not change. It is obvious that  $P \cdot P < 0$ , but  $P' \cdot P' = 2lH - H^2 - E^2 \cos^2[\omega(t - x)]$ . Thus if we choose  $l = H + \frac{E^2}{2H} \cos^2[\omega(t - x)]$ , the vector  $P'$  will be timelike,  $P' \cdot P' = H^2 > 0$ , and we can take  $P'/H$  as a properly normalized monad,

$$\tau = \left( 1 + \frac{E^2}{H^2} \cos^2[\omega(t - x)] \right) (dt - dx) + dx + \frac{E}{H} \cos[\omega(t - x)]dz. \quad (11.12)$$

Now the dually conjugated field tensor reads  $*F = -H\tau \wedge (dt - dx)$ , thus in the frame  $\tau$  the electric field (3.4) vanishes, and this is the field's co-moving frame. In all these calculations one has to remember that when only one (here, magnetic) field survives after the reference frame is transformed, there



are other possible transformations which do not change this situation (in fact, all those which involve an additional motion in the direction of this field, even when this motion occurs to be with a non-constant magnitude of the three-velocity described by strictly local Lorentz transformations working in non-inertial frames). Thus there appears a continuum of such one-field frames (*cf.* [7], but working in general as well as in special relativity), and the search for more elegant ones depends on the individual taste of the researcher.

Let us now calculate the three-velocity of the frame  $\tau$  from the viewpoint of  $\tau_{\text{in}}$  using our general definition (3.8) and substitute the result into the left-hand side of (5.1), then putting into the right-hand side the expressions of  $\mathbf{E}$  and  $\mathbf{B}$  from (11.10) in the frame  $\tau_{\text{in}}$  to check if the Landau–Lifshitz definition (5.1) really works. Obviously, this way will not represent a vicious circle since these parts of (5.1) were initially deduced in [7] from a very different standpoint than ours (moreover, in this way the left-hand side of (5.2) will be automatically checked: both definitions of  $\mathbf{v}$  cannot simultaneously work well). First, we rewrite (3.8) in these notations for frames and find  $\mathbf{v}$  ( $\perp \tau_{\text{in}}$ ):

$$\tau = (\tau \cdot \tau_{\text{in}})(\tau_{\text{in}} + \mathbf{v}) \Rightarrow \mathbf{v} = \frac{\frac{E}{H} \cos[\omega(t-x)]}{1 + \frac{E^2}{H^2} \cos^2[\omega(t-x)]} \left( dz - \frac{E}{H} \cos[\omega(t-x)] dx \right).$$

This means that

$$\frac{|\mathbf{v}|}{1 + \mathbf{v}^2} = \frac{\frac{E}{H} \cos[\omega(t-x)] \sqrt{1 + \frac{E^2}{H^2} \cos^2[\omega(t-x)]}}{1 + 2 \frac{E^2}{H^2} \cos^2[\omega(t-x)]}.$$

Precisely the same is the result of calculating  $\frac{|\mathbf{E} \times \mathbf{B}|}{\mathbf{E}^2 + \mathbf{B}^2}$  — Landau and Lifshitz’s definition wins. (Pauli’s definition (5.2) cannot contain on its left-hand side the construction  $1 + 2 \frac{E^2}{H^2} \cos^2[\omega(t-x)]$  which inevitably appears on the right-hand side  $2 \frac{|\mathbf{E} \times \mathbf{B}|}{\mathbf{E}^2 + \mathbf{B}^2}$ , like in the Landau–Lifshitz case.) The mean value of  $|\mathbf{v}|$  is simply  $\frac{2}{\pi} \arcsin \frac{E}{\sqrt{E^2 + H^2}}$ . When  $H \rightarrow 0$ , the mean propagation velocity approaches that of light, while if  $E \ll H$ , the mean velocity can become as low as one wishes: to this end, it is necessary to use as strong magnetic field  $H$  as possible and/or choose a low-intensity wave in the superposition.

## 12 Concluding remarks

The results obtained in this paper are based on three simple observations: that the physical classification of electromagnetic fields should be formulated

using the properties of only two well known invariants of these fields, the complete description of reference frame is related only to the state of motion of a continuous multitude of test observers, and that the duality rotation (in the vein of Rainich–Misner–Wheeler, but in a more modern and general form) applied to a seed solution of Maxwell’s equations, yields a new solution in the same four-geometry which was generated by the seed solution *via* Einstein’s equations. We have proven that these suppositions really work together, and the duality rotation permits to *construct* qualitatively new solutions, belonging also to other desired types of electromagnetic fields in accordance with our classification. There is only one restriction separating the pure null type fields from those of other five types. The pure null type does not change under the duality rotation, becoming in fact the same solution of this pure type, though corresponding to another reference frame and displaying the Doppler effect in its generalized form also considered in this paper. As illustrations of application of our approach we discuss concrete examples of the Kerr–Newman (KN) solution and the Liénard–Wiechert (LW) field (to show the efficiency of our method also in special relativity). Moreover, we deduce three qualitatively new types of electromagnetic field creating the same four-geometry as the seed KN solution, thus describing other kinds of KN-like black holes. Studying the LW field, we come upon a new conclusion that the linearity of Maxwell’s equations does not automatically mean that different constituent parts of this field can be properly interpreted separately. Other characteristics of the field (such as the energy density and Poynting vector) have non-linear nature, thus a study of these characteristics constructed only of one or another parts of the LW solution, with omission of the combination (“interaction”) of these parts, means a disregard of important physical properties of the field, in particular, of its true propagation velocity. We have explicitly shown that this velocity of the complete LW solution is less than that of light, and we have given the physically full-fledged frame co-moving with LW field in which its Poynting vector exactly vanishes everywhere outside the world line of the source of this field (strangely, this fact was never noticed before). The last, but not least example is related to a simple superposition of two exact solutions of special-relativistic Maxwell’s equations, plane electromagnetic wave and homogeneous magnetic field in a vacuum. We show that this superposition, being itself an exact solution, always propagates with the velocity lesser than that of light, and we show that the elementary expression for this velocity is properly defined in [7], but not in [13]. (I must admit that at first I liked the definition given in Pauli’s

book much more than Landau–Lifshitz’s one: see, *e.g.*, [12].)

Finally, may I express my hope (to a certain extent, against hope), that the given here examples should lead our community of physicists to a more profound consideration of the non-trivial concept of reference frame and to its better understanding as a more physical than purely mathematical subject and an important ingredient in the description of physical reality. To console those who cannot accept the representation of reference frames through monads and Cartan’s forms, I would add that they can take instead any system of coordinates whose  $t$ -coordinate lines coincide with those of the  $\tau$ -congruence (the choice of spatial coordinates does not matter). In such a system, there will be realized precisely the same picture, though mathematics will feel awkward, while reference frames will seem to be silenced.

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